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Admissible target paths in economic models

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In this paper we answer the question which dynamic target paths can be achieved in an economy described by a linear time-varying difference equation. Additionally, we discuss the advantages and disadvantages of various policies that yield tracking of those paths. These problems are of great relevance for the theory of economic policy (e.g., in optimal cost problems and the tracking of equilibrium paths). Usually, economic models can be controlled appropriately by just simple, robust controllers which facilitate a short-period adaptation with respect to new information. Therefore, we study the design problem of stabilizing minimum variance controllers in more detail. The design of appropriate weights for the controllers is related to the available future knowledge on the economy.

1. Introduction

Many (economic) papers have discussed the problem of target path controllability (also called dynamic path controllability or perfect controllability). This problem concerns a prespecified time interval. The question is whether any time path of target variables can be attained in this interval by means of an appropriate choice of the policy instruments.

For linear time-invariant systems this problem has been studied in Aoki et al. (1975, 1979), Brockett et al. (1965), Buiter (1979), Maybeck (1982), Preston et al. (1972, 1974, 1982), Tinbergen (1952), and Wohltmann et al. (1981, 1983, 1984); for linear continuous time-varying systems in Albrecht et al. (1986), Grasse (1986), and Wohltmann (1985); for linear discrete time-varying systems in Engwerda (1988b, d); and for nonlinear discrete time-varying systems in Nijmeijer (1989).

From a policy point of view, the target path controllability property of an economy is a very nice one. For, anything a policy maker wishes to achieve

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in the prescribed time interval can be realized: the designer can always suit the policy maker. If the economy is not target-path-controllable, then a natural question is: which target paths can still be tracked. This question plays, e.g., a crucial role in determining the 'best' approximation of an economically desired output path and in the problem of tracking equilibrium target paths [see Buiter et al. (1981)]. Though this seems a reasonable question, the optimal control literature hardly pays any attention to it. Usually a target path is prescribed without any consideration of the question whether it can be obtained. For instance, the problem to design an optimal policy is often formulated as a minimization problem of a quadratic cost criterion, which expresses that deviations from prescribed reference paths are quadratically penalized both for target variables and for instruments. To take account of the fact that aims may conflict, different weights are then attached to variables to express the policy maker's preferences [see, e.g., Pindyck (1973), Turnovsky (1977), de Zeeuw (1984)]. However, the fact that it might happen that irrespective of the pursued policy the reference paths cannot be attained is often overlooked.

For that reason we study in this paper the question which target paths can be tracked given some abstract dynamic economic model, devoid of specific economic content. Any target path that can be tracked, in some sense, is called admissible. To incorporate also economic models which are subject to (possibly unanticipated) structural variation we assume that our model is described by a linear time-varying difference equation.

A subsequent question that arises for this kind of models is to design a control policy that succeeds in tracking an admissible target path, a so-called successful control. We discuss a number of successful controllers. Special attention is paid to their robustness w.r.t. (i) modeling inaccuracies, (ii) the required information about the evolution of the system parameters in the future, and (iii) exact knowledge of the present state of the system. In particular we discuss a number of controllers based on the optimization of a quadratic cost criterion with a one-time-period-ahead planning horizon. This choice is made because it was clearly demonstrated by Kydland et al. (1977) that a multi-planning period will be anticipated by rational economic agents, so that the proposed policy may become suboptimal.

To familiarize the reader with the problem sketched above we continue this introduction with a simple well-known example from macro-economics. It concerns the accelerator model introduced by Samuelson (1939).

Samuelson assumed that the following relations hold for a country whose economy is not significantly influenced by foreign countries:

(a) The national income (denoted by Y) is the sum of consumption (C), investment (I), and government expenditures (G) at any year k. That is: $Y(k) = C(k) + I(k) + G(k)$.

(b) The consumption is a fraction (b) of national income in the previous year. That is: $C(k) = bY(k-1)$, $b > 0$.

(c) The induced investment is a fraction (d) of the difference between the consumption now and consumption in the previous year. That is: $I(k) = d(C(k) - C(k-1))$, $d > 0$.

Some rewriting of the above relationships yields that national income, consumption, and government expenditures are related in the following way:

$$\begin{pmatrix} Y(k) \\ C(k) \end{pmatrix} = \begin{pmatrix} (1+d)b & -d \\ b & 0 \end{pmatrix} \begin{pmatrix} Y(k-1 \\ C(k-1) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} G(k) . \tag{1}$$

Now let Y^* and C^* be target values for income and consumption, respectively. Starting at time $k-1$ with initial values $\bar{Y}(k-1)$ and $\bar{C}(k-1)$, it is easily verified that the policy

$$\begin{pmatrix} G(k+1) \\ G(k) \end{pmatrix} = \begin{pmatrix} 1 & (1+d)b \\ 0 & b \end{pmatrix}^{-1} \times \left(\begin{pmatrix} Y^* \\ C^* \end{pmatrix} - \begin{pmatrix} (1+d)b & -d \\ b & 0 \end{pmatrix}^2 \begin{pmatrix} \bar{Y}(k-1) \\ \bar{C}(k-1) \end{pmatrix} \right)$$

achieves attainment of the targets at time $k+1$.

So, whatever the initial state

$$\begin{pmatrix} \bar{Y}(k-1) \\ \bar{C}(k-1) \end{pmatrix}$$

of system (1) is, there always exists a policy such that any target can be attained. This property of system (1) is known in the literature as (pointwise) controllability.

Once the target

$$\begin{pmatrix} Y^* \\ C^* \end{pmatrix}$$

is reached, the question can be raised whether there exists a policy such that from time $k+1$ on both income and consumption grow smoothly. That is, $Y(t+1) = \alpha Y(t)$ and $C(t+1) = \beta C(t)$ for some prespecified α and β from time $t = k+1$ on.

Simple calculation shows that such a policy only exists if the marginal propensity to consume (i.e., the structural parameter b) equals $\beta C(t)/Y(t)$ at any time $t > k + 2$. It will be clear that usually this does not occur. So, unless the structural parameters vary in time, it is not possible to track all prespecified target paths. That is, system (1) is not target-path-controllable [see also Engwerda (1988d)].

A natural question that now arises is, which growth paths can be tracked by our model (1). To study this phenomenon we first consider a country with no government interference, that is, we consider the system (1) with $G(t) = 0$, for all t. It is well-known that in that case, in the end, all variables of system (1) for almost all initial states of the system grow with a growth rate λ, where λ is the spectral radius of the system matrix

$$A := \begin{pmatrix} (1+d)b & -d \\ b & 0 \end{pmatrix},$$

i.e., $\max\{|\lambda_1|, |\lambda_2|\}$, where λ_1 and λ_2 are the eigenvalues of matrix A. (Only for those initial states which are contained in the eigenspace corresponding with the eigenvalue λ_i, which is in norm smaller than the spectral radius, the growth rate is smaller.)

Moreover, since generically all eigenvalues of a matrix differ, the eigenspace corresponding to the eigenvalue of largest modulus will be a one-dimensional space. So without government interference, generically, income and consumption grow in the end at the same rate and at a fixed ratio.

To study the impact of government interference we reconsider system (1). Assume that at time k government expenditures are once raised by one unit. The effect of such an increase is a shift of the growth paths for both income and consumption,[1] whereas the growth rates remain unchanged.

A different feasible control policy determines government expenditures as a fixed mixture of previous income and consumption at any given time, i.e.,

$$G(t) = F\begin{pmatrix} Y(t-1 \\ C(t-1) \end{pmatrix} \quad \text{for all} \quad t > k.$$

This type of policy is capable of changing the spectral radius of the system matrix. That is, we can manipulate the growth rate of the system. As an example take, e.g., $F := (-(1+d)b + 1.04 | -d)$. This choice will lead to a 4% annual growth rate of both income and consumption.

In conclusion, we have that by a careful chosen mixture of both policies we can manipulate the growth rate of income and consumption as well shift

[1] Note that a shift here does not mean that the whole trajectory is shifted at any time with the same amount. A shift of, e.g., the income trajectory $Y(.)$ should in this context be interpreted in the following sense: due to a change in the initial state another trajectory $\tilde{Y}(.)$ is followed which has the property that either $\tilde{Y}(k) > Y(k)$ for all k or $\tilde{Y}(k) < Y(k)$ for all k.

these growth paths upwards or downwards. However, it is not possible to attain different robust smooth growth rates for the income and consumption trajectories.

We hope this example convinced the reader that arbitrarily prescribed growth of different economic variables in general is not possible, even when a policy maker is not restricted in the use of instruments. Moreover, we hope that this motivates the study of the central issues of this paper, namely the characterization of admissible target paths and the construction of successful robust controllers.

2. Definitions and notation

The basic system analyzed in this paper is described by the following linear, finite-dimensional, time-varying difference equation:

$$\Sigma_{yd}: \begin{array}{l} x(k+1) = A(k)x(k) + B(k)u(k) + G(k)d(k), \quad x(k_0) = x_0\,, \\ y(k) = C(k)x(k)\,, \end{array}$$

where $x(k)$ is an n-dimensional vector consisting of endogenous variables (state variables) observed at time k; $u(k)$ is an m-dimensional vector of policy instruments (control variables); $d(k)$ is an s-dimensional vector of deterministic (noncontrollable) variables, assumed to be known at time k; $y(k)$ is an r-dimensional vector of target variables; and x_0 is the initial state of the system. We assume that the system parameters [i.e., the matrices $A(k)$, $B(k)$, $G(k)$, and $C(k)$] are bounded in time. The results presented in this section and the next one either are known or straightforward to prove (by using elementary substitution and least squares approximation techniques), and so we omit the proofs. The following notation will be used:

$v^*(i)$ denotes a reference value for variable v at time i,

$v^{\mathrm{T}}(i)$ denotes the transpose of $v(i)$,

$v[k, l] := (v^{\mathrm{T}}(k), \ldots, v^{\mathrm{T}}(l))^{\mathrm{T}}$,

$v[k, .] := (v^{\mathrm{T}}(k), v^{\mathrm{T}}(k+1), \ldots)^{\mathrm{T}}$,

$A^+ :=$ Moore–Penrose inverse of A,

$A(k+i, k) := A(k+i-1) \times \cdots \times A(k)$ for $i > 0$ and $A(k, k) := I$,

$S[k, k-N] := [B(k) \,|\, A(k+1, k)B(k-1) \,|\, \ldots \,|\, A(k+1, k-N+1)B(k-N)]$,

$V[k, k-N] := [G(k) \,|\, A(k+1, k)G(k-1) \,|\, \ldots \,|\, A(k+1, k-N+1)G(k-N)]$.

A matrix that plays a crucial role in this paper is the impulse response matrix $M(k; p, q)$ from $u[k+p+q-2, k]$ to $y[k+p+q-1, k+p]$, which is well-known in engineering. Let $t = k + p$. Then $M(k; p, q)$ is by definition equal to

$$\begin{bmatrix} C(t+q-1)B(t+q-2) & C(t+q-1)A(t+q-2)B(t+q-3) & \dots & C(t+q-1)A(t+q-1,k+1)B(k) \\ 0 & & & \\ \vdots & & & \\ 0 \dots 0 & C(t)B(t-1) \; C(t)A(t-1)B(t-2) & \dots & C(t)A(t,k+1)B(k) \end{bmatrix}$$

This matrix tells us what the effect of the policy instruments is on the target variables, that is, when $x_0 = 0$ and $d(.) = 0$ we have $y[k_0 + p + q - 1, k_0 + p] = M(k_0; p, q)u[k_0 + p + q - 2, k_0]$. Usually p is called the policy lead and q is called the target interval.

Note that this matrix also plays a fundamental role in the target path controllability problem and in the tracking problem. In the first case this matrix must have full row rank for some p, q, whereas in the second case this rank condition is not necessary [see, e.g., Engwerda (1988d, lemma 8, theorem 8)].

3. Characterization of admissible trajectories

3.1. Formal definitions and results

In the introduction we motivated the study of target paths that can be tracked by system Σ_{yd}. The notion of tracking was, however, not precisely defined. In fact the notion of tracking can be interpreted in various ways. We will distinguish three senses in which a target path can be tracked.

In the first sense, a target path can be tracked exactly during a prespecified time interval. Any trajectory that has this property is called strongly-admissible. The second possibility is that a trajectory can be tracked exactly up to an error not exceeding α over a prespecified time interval. A target path that can be tracked in this sense is called approximately-admissible of level α. Finally, a target path can be tracked asymptotically. Any target path that has this property is called asymptotically-admissible. The formal definitions read as follows.

Definition 0. *Let* $-\infty < k_0 \leq k < l \leq \infty$, *and let* x *be the state of* Σ_{yd} *at* k_0. *A reference trajectory* $y^*[k, l]$ *is called*

– *strongly-admissible for* x *at* k_0 *if there exists a control sequence* $u[k_0, l-1]$ *such that* $y[k, l] = y^*[k, l]$;

– *approximately-admissible of level* α *for* x *at* k_0 *if there exists a control sequence* $u[k_0, l-1]$ *such that* $\|y[k, l] - y^*[k, l]\| \leq \alpha$.

A reference trajectory $y^*[k_0, .]$ *is called*

– *asymptotically-admissible for* x *at* k_0 *if there exists a control sequence* $u[k_0, .]$ *such that* $\|y(k) - y^*(k)\| \to 0$ *for* $k \to \infty$.

In each of these three cases we call $u(.)$ *a successful control.* ■

In the remainder of this subsection we derive a necessary and sufficient condition on the dynamic evolution of an admissible target path, a criterion to check whether a prespecified trajectory is admissible, and a criterion for the success of controllers for strongly- and approximately-admissible target paths.

The characterization of admissible target paths reads as follows:

Theorem 1. A reference trajectory $y^*[k, l]$ *is*

A: *strongly-admissible for* x *at* k_0 *iff there exists a* $u[k_0, l-1]$ *such that*

$$y^*(t) = C(t)x^*(t), \quad k \leq t \leq l,$$

$$x^*(t+1) = A(t)x^*(t) + B(t)u(t) + G(t)d(t), \quad x^*(k_0) = x, \quad k_0 \leq t \leq l-1;$$

B: *approximately-admissible of level* α *for* x *at* k_0 *iff there exist* Δ, $u[k_0, l-1]$, $v[k_0, l-1]$, *and* $w[k_0, l-1]$ *such that*

$$y^*(t) = C(t)x^*(t) - w(t), \quad k \leq t \leq 1,$$

$$x^*(t+1) = A(t)x^*(t) + B(t)u(t) + G(t)d(t) - v(t), \quad x^*(k_0) = x - \Delta,$$

$$k_0 \leq t \leq l-1,$$

$$\text{and } \sum_{i=k}^{l} \|C(i)A(i, k_0)\Delta + \sum_{j=k_0+1}^{i} C(i)A(i,j)v(j-1) + w(i)\| \leq \alpha;$$

C: *asymptotically-admissible for* x *at* k_0 *iff there exist* $u[k_0, .]$ *and* $v[k_0, .]$, *where* $v(.) \to 0$, *such that*

$$y^*(t) = C(t)x^*(t) + v(t),$$

$$x^*(t+1) = A(t)x^*(t) + B(t)u(t) + G(t)d(t), \quad x^*(k_0) = x, \quad t \geq k_0. \quad ■$$

Note that, due to the fact that command errors can compensate each other when time passes, part B of this theorem is much more involved than part A.

In order to study how admissibility of a prespecified reference trajectory can be checked, we introduce the notion of command error flow. The command error flow is defined as the difference between the flow of Σ_{yd} from k_0 up to k (with control variables identically zero) and the target reference value at k. Formally:

Definition 2. The command error flow at k from k_0, $z_{k_0}(k)$, is defined as

$$z_{k_0}(k) := y^*(k) - C(k)A(k, k_0)x(k_0) - C(k)V[k-1, k_0]d[k-1, k_0].$$

The criteria to verify admissibility of a prespecified target path are now the following:

Theorem 3. A reference trajectory $y^*[k_0 + p, k_0 + p + q - 1]$ is

A: *strongly-admissible for x at k_0 iff*

$$\|(M(k_0; p, q)M^+(k_0; p, q) - I)z_{k_0}[k_0 + p + q - 1, k_0 + p]\| = 0;$$

B: *approximately-admissible of level α for x at k_0 iff*

$$\|(M(k_0; p, q)M^+(k_0; p, q) - I)z_{k_0}[k_0 + p + q - 1, k_0 + p\| \leq \alpha.$$

A reference trajectory $y^*[k_0, .]$ is

C: *asymptotically-admissible for x at k_0 iff*

$$\lim_{k \to \infty} \|\{(C(k)S[k-1, k_0])(C(k)S[k-1, k_0])^+ - I\}z_{k_0}(k)\| = 0. \quad \blacksquare$$

From this theorem it is moreover easily seen that the least-squares solution $u[k_0 + p + q - 2, k_0] = M^+(k_0; p, q)z_{k_0}[k_0 + p + q - 1, k_0 + p]$ is a successful control for both strongly- and approximately-admissible target paths. It is also clear from this theorem that for a given lead p a minimal α exists for which a trajectory is approximately-admissible of level α. This α is given by (2) where the inequality has to be replaced by an equality.

3.2. Interpretation of the results

Theorem 1 leads to the intuitively very appealing result that any admissible reference trajectory must satisfy a recurrence equation which corresponds to the given system. The only difference with the system is that the reference trajectory may possess an additional disturbance. The additional disturbance must satisfy some properties which depend on the kind of admissibility that is considered. They are more stringent for the strongly- and approximately-admissible trajectories than for the asymptotically-admissible ones. So, it is more difficult to find an strongly- or approximately-admissible target path than an asymptotically-admissible one. We will see in section 4.1 that this ordering in difficulty carries over to the problem of constructing a successful control.

Concerning the suggested successful open-loop control we note that it uses all information of the deterministic variables and reference trajectories in the time interval $[k_0, k_0+p+q-1]$, and that this information must be available already at time k_0. Engwerda (1988b, p. 60) showed in an example that there exist situations in which this information really cannot be missed in the design of a successful control. Since information about the future development of deterministic variables is rarely known, the design of a successful control becomes a delicate matter in those cases.

We conclude this section with an application of the developed theory to the tracking of equilibrium paths in the Samuelson model (1).

First, we recall that a state is called an equilibrium of system (1), whenever there exists a level of government expenditures $\bar{G}$ such that the state of the system does not change if $\bar{G}$ is applied. So the state $\binom{Y^*}{C^*}$ is an equilibrium if and only if the corresponding target trajectories $Y^*[p,p+1] := (Y^*, Y^*)^{\mathrm{T}}$ and $C^*[p,p+1] := (C^*, C^*)^{\mathrm{T}}$ are strongly-admissible for a lead of two. Application of Theorem 3.A yields that this holds if and only if $C^* = bY^*$. Another consequence, which is immediately clear from model (1), is that in equilibrium there is no net investment, i.e., $I(.) = 0$.

Next, we want to determine the approximately-admissible trajectories at a minimal level α. For this purpose, we assume that $Y^*[0,.] := (Y^*, Y^*, ..)^{\mathrm{T}}$ and $C^*[0,.] := (C^*, C^*, ..)^{\mathrm{T}}$ are the desired equilibrium target paths, $\binom{\bar{Y}}{\bar{C}}$ is the initial state of the system, and the considered lead equals one. Elementary calculation gives the next result for the open-loop successful control $u[k-1,0] = M^+(0;1,k)z_0[k,1]$:

$$u[0,k-1] := \left[\frac{b}{1+b^2}C^* + \frac{1}{1+b^2}Y^* - b(1+d)\bar{Y} + d\bar{C},\right.$$

$$\frac{b-(1+d)b^2}{1+b^2}C^* + \frac{1-(1+d)b}{1+b^2}Y^* + bd\bar{Y},$$

$$\frac{b(1-b)}{1+b^2}C^*, \ldots, \frac{b(1-b)}{1+b^2}C^*,$$

$$\left.\frac{b^2-b+1}{1+b^2}Y^* - \frac{b^2}{1+b^2}C^*\right].$$

Application of this control to system (1) then yields the output trajectories

$$Y[1,k] := \left[\frac{b}{1+b^2}C^* + \frac{1}{1+b^2}Y^*, \dots, \frac{b}{1+b^2}C^* + \frac{1}{1+b^2}Y^*, \right.$$
$$\left. \frac{1-b^2}{1+b^2}C^* + Y^*\right],$$

$$C[1,k] := \left[b\bar{Y}, \frac{b^2}{1+b^2}C^* + \frac{b}{1+b^2}Y^*, \dots, \frac{b^2}{1+b^2}C^* + \frac{b}{1+b^2}Y^*\right].$$

This clearly demonstrates how equilibrium paths in the Samuelson model are tracked optimally, in the sense that the total tracking error is minimized.

The first two control periods are used to regulate income and consumption to a feasible equilibrium at a minimal distance from the desired value (Y^*, C^*); in this case,

$$Y = \frac{b}{1+b^2}C^* + \frac{1}{1+b^2}Y^*, \qquad C = b\left(\frac{b}{1+b^2}C^* + \frac{1}{1+b^2}Y^*\right).$$

From time 2 up to $k-1$ a constant input, here

$$u(.) = \frac{b(1-b)}{1+b^2}C^*,$$

is applied next in order to stay at the equilibrium. Finally, the fact that time k is the final planning period facilitates an additional reduction of the command error in this period.

4. Characterization of admissible minimum-variance controlled target paths

4.1. The general case

In the previous section we gave a characterization of admissible target paths for a system and derived open-loop successful controllers. Disadvantages of these open-loop controllers are that they require much information about the future development of the system and that they are not robust.

Now, an economy is a typical example of a process in which the evolution of the system parameters in time is difficult to predict, and exact information concerning the state of the system is often lacking. Moreover, rational economic agents may react to economic policy if this is the outcome of a planning procedure with a time horizon larger than one [see Kydland et al.

(1977)]. Therefore, there is a need for simple robust successful controllers which facilitate a short-period adaptation of control with respect to new information.

A class of controllers that has these properties and that is moreover optimal, in the sense that it minimizes a prespecified cost criterion, is the class of Minimum Variance (MV) controllers [see, e.g., Åström (1983, 1984), Aalders et al. (1983), and Engwerda (1988c)]. For that reason we will look for successful controllers of this type for the system Σ_{yd} with added white-noise disturbances.

So, we consider the system

$$\tilde{\Sigma}_{yd}: \begin{array}{l} x(k+1) = A(k)x(k) + B(k)u(k) + G(k)d(k) + \mu(k), \quad x(k_0) = \tilde{x}, \\ y(k) = C(k)x(k) + v(k), \end{array}$$

where $\mu(k)$ and $v(k)$ are independent white-noise vectors with $\mathrm{cov}\{\mu(k)\mu^{\mathrm{T}}(l)\} = \Sigma_{\mu}\delta_{kl}$ and $\mathrm{cov}\{v(k)v^{\mathrm{T}}(l)\} = \Sigma_{v}\gamma_{kl}$. Here δ_{kl} is the Kronecker delta which equals 1 if $k = 1$ and 0 otherwise.

Simple calculation shows that Theorem 1.C remains valid in the sense that it gives a necessary and sufficient condition for convergence of the error $\mathrm{E}\{y(k) - y^*(k)\}$ to zero subject to $\tilde{\Sigma}_{yd}$.

Formally, the MV controller is then defined as the controller which at any time k minimizes the following cost criterion, J, subject to the system $\tilde{\Sigma}_d$ [i.e., system $\tilde{\Sigma}_{yd}$ with $y(k) = x(k)$ for all k]:

$$J := \mathrm{E}\{(x(k+1) - x^*(k+1))^{\mathrm{T}} Q(k+1)(x(k+1) - x^*(k+1))\},$$

for a given initial state $\tilde{x}$ and reference trajectory $x^*(.)$. Here $Q(k)$ is a positive semi-definite matrix.

The cost criterion expresses that positive and negative deviations of target variables from desired levels are weighted equally and quadratically. Moreover, the matrix Q can be used to express preferences. In particular, past responses of economic agents to the applied policy can be incorporated by rescaling the weight matrix chosen earlier.

It is well-known [see, e.g., Preston et al. (1982)] that

$$\begin{aligned} u(k) = &-(B^{\mathrm{T}}(k)Q(k+1)B(k))^{+} B^{\mathrm{T}}(k)Q(k+1) \\ &\times (A(k)x(k) + G(k)d(k) - x^*(k+1)) \end{aligned} \tag{3}$$

is a minimizing solution to this problem.

Moreover, it is clear that application of the MV controller yields, with $e(k) := x(k) - x^*(k)$, the closed-loop error equation

$$
\begin{aligned}
e(k+1) &= P(k)A(k)e(k) + \mu(k), \quad e(0) = \bar{e}(0), \\
y(k) - y^*(k) &= C(k)e(k) - w(k) + v(k),
\end{aligned}
\tag{4}
$$

where $w(.)$ converges to zero, $\bar{e}(0)$ may be unequal to zero, and $P(k)$ denotes the matrix $I - B(k)(B^{\mathrm{T}}(k)Q(k+1)B(k)^{+} B^{\mathrm{T}}(k)Q(k+1)$. So, the expected command error, $\mathrm{E}\{y(k) - y^*(k)\}$, converges to zero when $k \to \infty$. Of course this fact does not give much information if the covariance of this error grows to infinity. Therefore the question arises under which conditions this error remains bounded. The next theorem deals with an important case.

Theorem 4. Let $Q(.)$ be such that $(PA)(.)$ is uniformly exponentially stable [see, e.g., Willems (1970) for a definition of this notion]. Then $y^*(.)$ is asymptotically admissible w.r.t. $\tilde{\Sigma}_{yd}$ iff (if and only if) $y^*(.)$ satisfies the conditions of Theorem 1.C. Moreover, the covariance matrix of the error $(y - y^*)(.)$ remains bounded if the MV controller with weighting matrix $Q(.)$ is used to control the system $\tilde{\Sigma}_{yd}$. ■

In the sequel we will call $Q(.)$ a stabilizing weighting matrix sequence if it is such that $(PA)(.)$ becomes uniformly exponentially stable. So an interesting question is under which conditions there exists a stabilizing weighting matrix sequence $Q(.)$. In this general form, this is a difficult problem to solve. Therefore, we just state a sufficient condition for it in Construction 3 of section 4.2 below. For time-invariant systems (i.e., for systems with the property that the structural parameters do not change in time) things become less involved.

Theorem 5. Let $\tilde{\Sigma}_{yd}$ be time-invariant. Then there exists a stabilizing weighting matrix Q iff the matrix pair (A, B) is stabilizable [see, e.g., Preston et al. (1982) for a definition].

Proof. It is obvious that the condition is necessary. The sufficiency can be proved according to the lines of the proof of proposition 1 in Engwerda et al. (1989). ■

4.2. Choice of weight matrix $Q(k+1)$

In this section, we discuss some advantages and disadvantages of several choices of the weighting matrix $Q(k+1)$ in the MV controller. First, we discuss the construction of a stabilizing weight matrix if we know very much about the development of the structural parameters $A(.)$ and $B(.)$ in time and, later on, constructions which are based on less information concerning these parameters are considered.

The first special case we consider is a time-invariant controllable system, i.e., a system with the property that the matrix $S[0, n-1]$ has full row rank.

Construction 1. *Let (A, B) be controllable. Then there exists a weighting matrix Q_1 such that PA is nilpotent with index κ_1 (κ_1 is the controllability index; it is a natural number which is smaller than n). That is, there exists a weighting matrix Q_1 such that $(PA)^{\kappa_1} = 0$. Moreover, this weighting matrix equals S^TS, where S is a transformation matrix which transforms (A, B) into its Luenberger phase canonical form $(\bar{A}, \bar{B})$ [see Luenberger (1967)].* ■

If system Σ_{yd} is not subjected to white-noise terms, then the MV controller u_1^0 based on this weighting matrix seems to be the best one. For, any initial error $e(0)$ is controlled to zero as quickly as possible. But note that this is not robust. No attention is paid to the amount of or fluctuations in the applied control. So, in general this controller responds most eagerly to any deviation of the nominal path from its reference path. As a consequence, a heavily fluctuating controller is obtained if a system containing white noise is controlled. This may be a reason for the policy maker not to use this controller. A weighting matrix that generically yields a smoother controller (in the sense that the controller is less fluctuating) is formalized in the next construction.

Construction 2. *Let Q_1 be the positive definite matrix obtained in Construction 1, (A, B)-stabilizable, and Q_2 the (unique) positive definite solution of the algebraic Riccati equation: $Q_2 = A^T\{Q_2 - Q_2B(B^TQ_2B)^{-1}B^TQ_2\}A + Q_1$. Then, Q_2 is a stabilizing sequence of weighting matrices.* ■

Note that the corresponding controller u_2^0 also minimizes the cost criterion

$$\sum_{k=1}^{\infty} e^T(k)Q_1e(k) \quad \text{subject to} \quad e(k+1) = Ae(k) + Bu(k)\,.$$

Denote this minimum value by $J(u_2^0)$. Then obviously $J(u_2^0) \leq J(u_1^0)$. So, applying controller u_2^0 is less costly than applying u_1^0.

We now construct a stabilizing weighting matrix sequence $Q(.)$ for a time-varying system. This construction uses the notion of uniform periodic smooth exponential stabilizability. This notion was introduced by Engwerda in (1988b) [see also Engwerda (1990a, b)]. It naturally generalizes the notion of stabilizability of a time-invariant system. Roughly speaking, a time-varying system Σ_{yd} is called uniformly periodically smoothly exponentially stabilizable if, firstly, any disturbance entering the controllable part of the system can be steered periodically to zero by means of a bounded input sequence (the smoothness property) and, secondly, the uncontrollable part of the system is uniformly exponentially stable. To guarantee that with the next choice of weighting matrices our MV controller has this smoothness property we additionally assume that matrix $B(k)$ satisfies at each time k some regularity condition. A rigorous proof is given in the appendix.

Construction 3. *Let $(A(.), B(.))$ be uniformly periodically smoothly exponentially stabilizable and assume, moreover, that there exists an $\beta > 0$ such that $B^{\mathrm{T}}(k)B(k) \geq \beta I$ for all $k \geq 0$. Then there exists a sequence of matrices $Q(.)$ such that for all k, $\lim_{N\to\infty} (PA)(k, N) = 0$. Here, at any point k in time, $Q(k)$ is obtained as $\lim_{N\to\infty} Q_N(k)$, where $Q_N(k)$ is the solution to the Riccati recurrence equation*:

$$\begin{aligned} Q_N(i) &= A^{\mathrm{T}}(i)\{Q_N(i+1) - Q_N(i+1)B(i)(B^{\mathrm{T}}(i)Q_N(i+1)B(i))^{-1} \\ &\quad \times B^{\mathrm{T}}(i)Q_N(i+1)\}A^{\mathrm{T}}(i) + I, \qquad (5) \\ Q_N(N) &= I. \quad \blacksquare \end{aligned}$$

In our last construction of a weighting matrix we assume that the future structural parameters are unknown to the policy maker and, moreover, that he has no exact information about the current state of the system. All he knows are the current system parameters.

Based on this information the most rational policy is to choose the weighting matrix such that the norm of the current closed-loop system matrix, $P(k)A(k)$, is minimized. Engwerda showed in (1988a) [see also Engwerda (1989)] the following result:

Construction 4. *If at time k only $A(k)$ and $B(k)$ are known to the designer, then the best choice (in the sense of optimal stabilization policy) for Q is the identity matrix I. In that case $\|P(k)A(k)\|_2$ [i.e., the largest singular value of matrix $P(k)A(k)$, also called the operator norm of this matrix] is as small as possible, although in general not smaller than one.* $\blacksquare$

Note that, with this choice of Q, it may happen that we obtain an unstable closed-loop system though the system is stabilizable (as might turn out after all).

We conclude this section by reconsidering the Samuelson model (1). Straightforward calculations show that the four weighting matrices for this model are

$$Q_1 = \frac{1}{(1+\alpha)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1/b^2 \end{pmatrix}, \quad Q_2 = \frac{1}{(1+\alpha)^2}\begin{pmatrix} 2 & 0 \\ 0 & 1/b^2 \end{pmatrix},$$

$$Q_3(k) = \begin{pmatrix} 1 + b^2(k) & 0 \\ 0 & 1 \end{pmatrix},$$

under the assumption that all parameters in (1) depend on time, and

$$Q_4 = I\,,$$

respectively.

It is interesting to note that all these weighting matrices yield the same control gain, i.e., the matrix $-(B^{\mathrm{T}}QB)^{-1}B^{\mathrm{T}}Q$ equals $\left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right)$ for all choices of Q. So, one might say that the Samuelson model is a robust model in the sense that it is not necessary to provide *a priori* estimates concerning the future development of the structural parameters in this model, since the optimal MV control strategies ignore them.

5. A simulation study

In this final section we illustrate the influence of the weighting matrix Q in the MV controller on its tracking performance. A small macro-economic model, estimated by Kendrick (1982) for the U.S. economy, serves as a starting point for the various experiments. The estimated reduced form reads as follows:

$$\begin{aligned}\begin{pmatrix}C(k)\\ I(k)\end{pmatrix} &= \begin{pmatrix}0.914 & -0.016\\ 0.097 & 0.424\end{pmatrix}\begin{pmatrix}C(k-1\\ I(k-1)\end{pmatrix}\\ &\quad + \begin{pmatrix}0.305 & 0.424\\ -0.101 & 1.459\end{pmatrix}\begin{pmatrix}u_1(k-1)\\ u_2(k-1)\end{pmatrix}\\ &\quad + \begin{pmatrix}-59.437\\ -184.766\end{pmatrix} + \begin{pmatrix}v_1(k)\\ v_2(k)\end{pmatrix},\end{aligned}$$

where $C(k)$ and $I(k)$ are as in section 1; $u_1(k)$ and $u_2(k)$ are governmental expenditures and money supply, respectively; $V^{\mathrm{T}}(k) := (v_1^{\mathrm{T}}(k)v_2^{\mathrm{T}}(k))$ is a white-noise vector with $\operatorname{cov}\{V(k)V^{\mathrm{T}}(s)\} = \left(\begin{smallmatrix}3.73 & 0\\ 0 & 8.58\end{smallmatrix}\right)\delta_{ks}$; and the initial values are $C(0) = 387.9$ and $I(0) = 85.3$.

Since Q has no influence as long as the matrix B is invertible, we only consider one control variable in the next experiments. In the first two experiments the chosen input is the money supply. In other words: we take $u_1(.)$ to be zero here. For the growth paths of consumption and investment we choose the following asymptotically admissible trajectories:

$$\begin{aligned}\begin{pmatrix}C^*(k+1)\\ I^*(k+1)\end{pmatrix} &= \begin{pmatrix}0.914 & -0.016\\ 0.097 & 0.424\end{pmatrix}\begin{pmatrix}C^*(k)\\ I^*(k)\end{pmatrix}\\ &\quad + \begin{pmatrix}0.424\\ 1.459\end{pmatrix}u_2^*(k) + \begin{pmatrix}-59.437\\ 184.766\end{pmatrix},\end{aligned}$$

with $C^*(0) = 480$, $I^*(0) = 100$, $u_2^*(k+1) = 1.015u_2^*(k)$, and $u_2^*(0) = 157.3$.

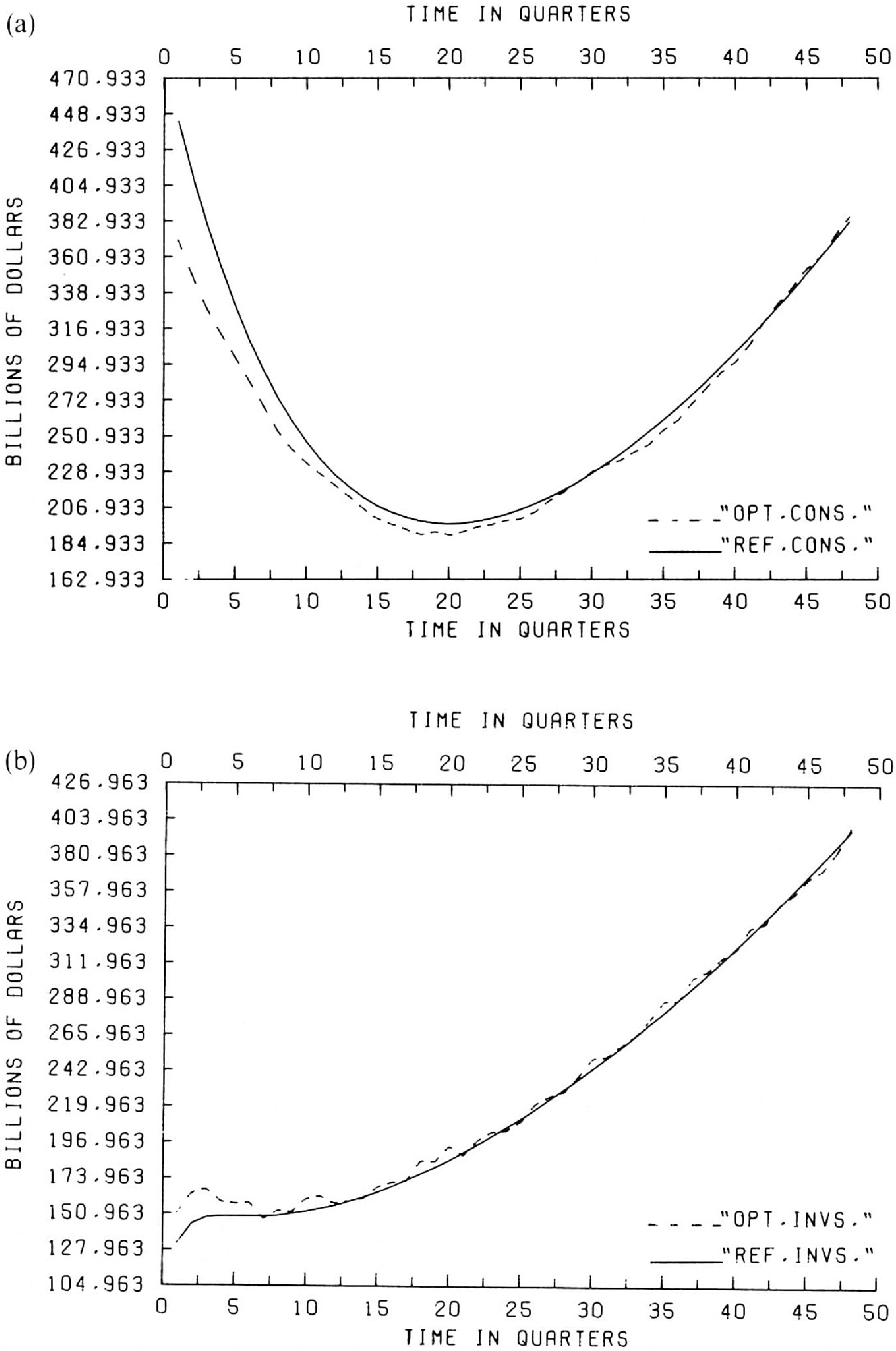
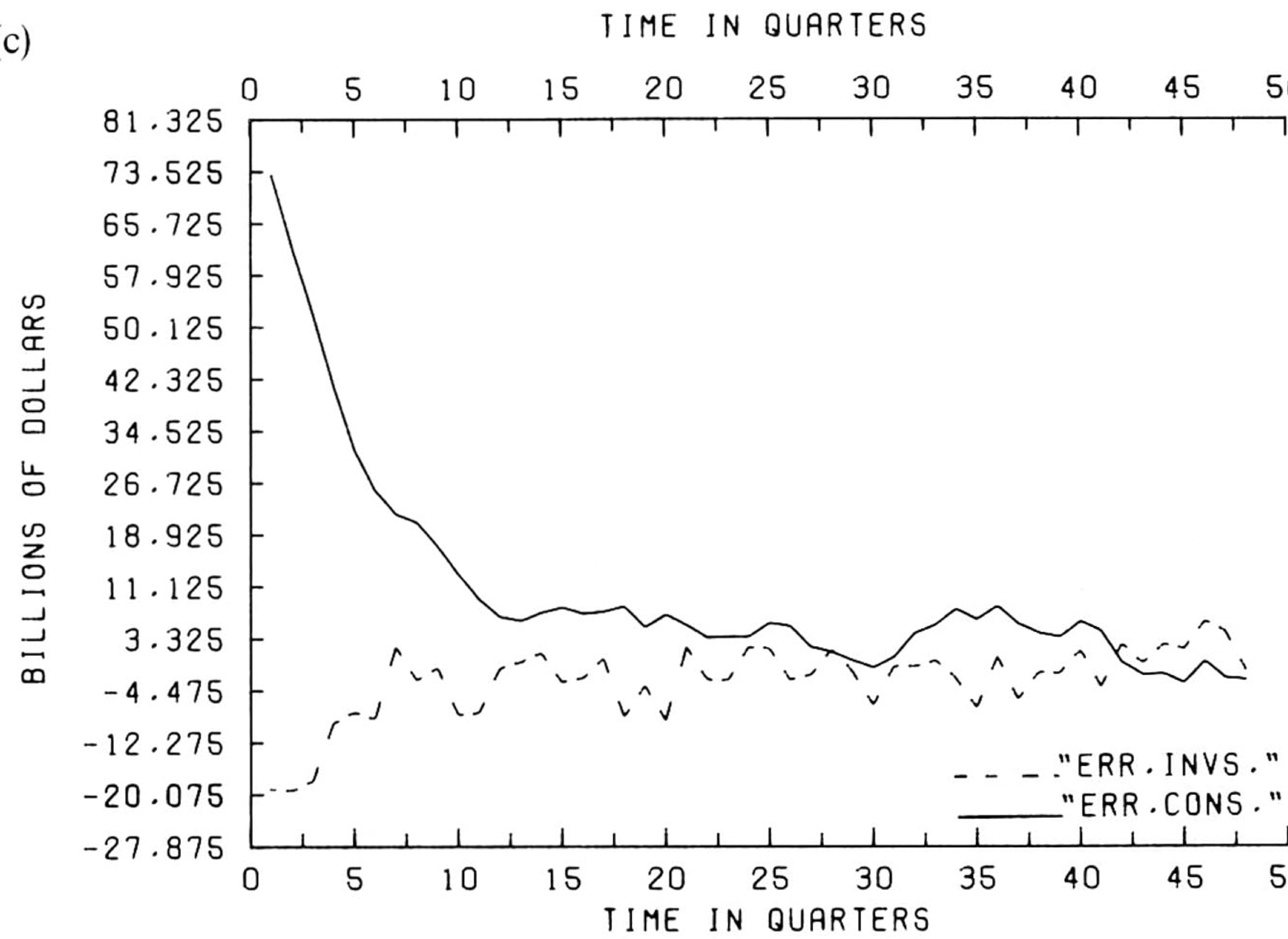


Fig. 1. Experiment 1.

(c)



(d)

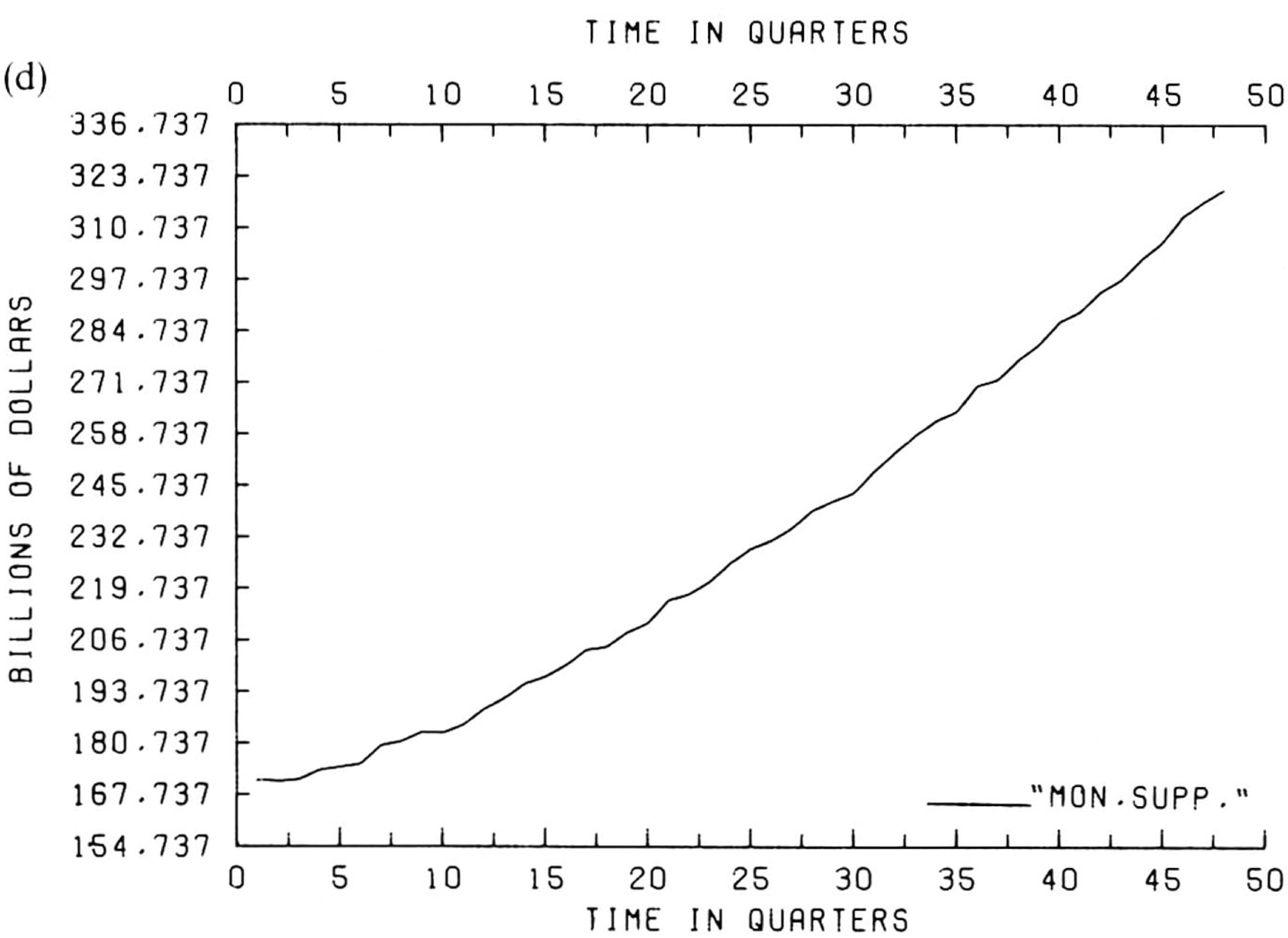


Fig. 1 (continued)

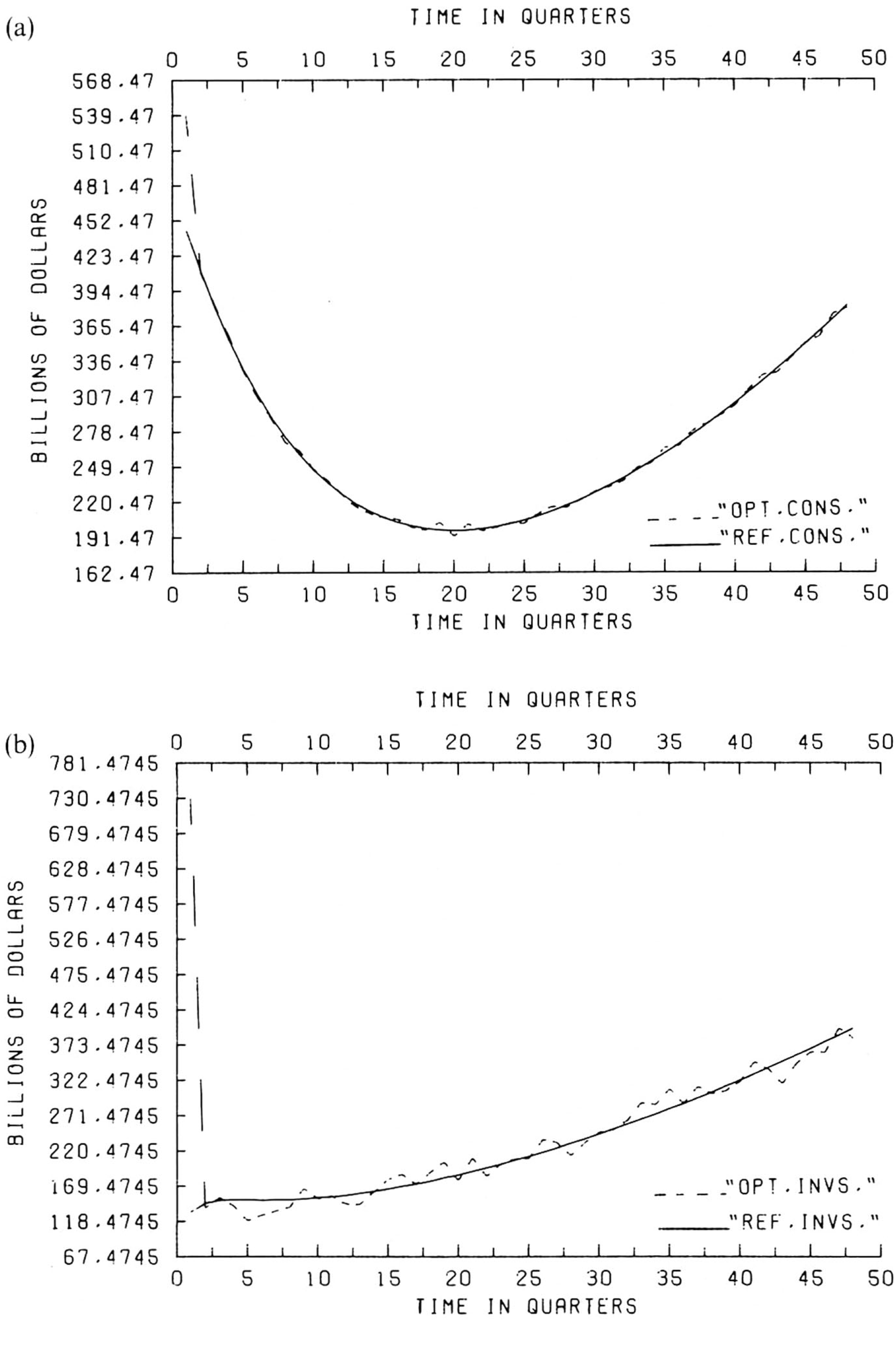
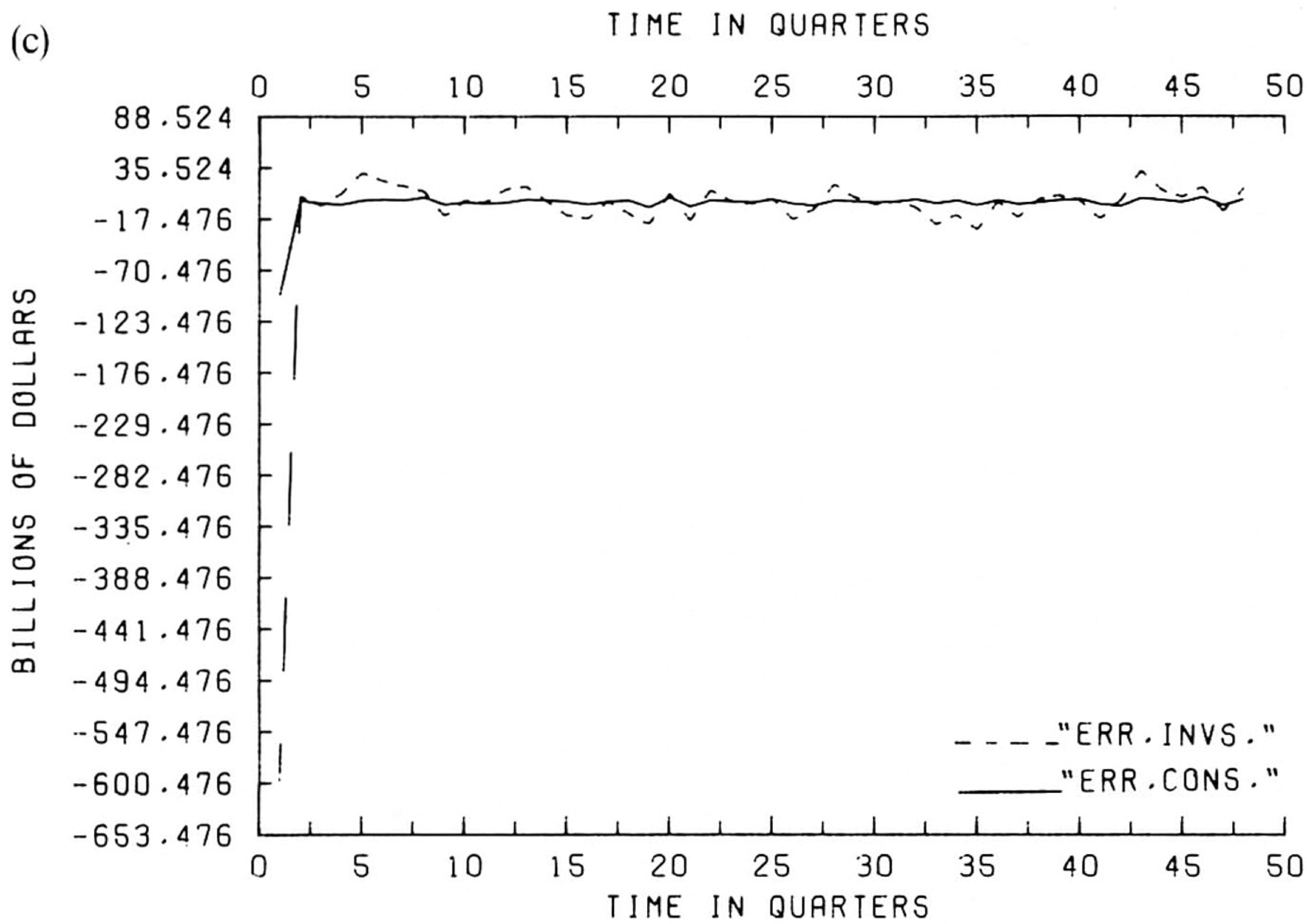


Fig. 2. Experiment 2.

(c)



(d)

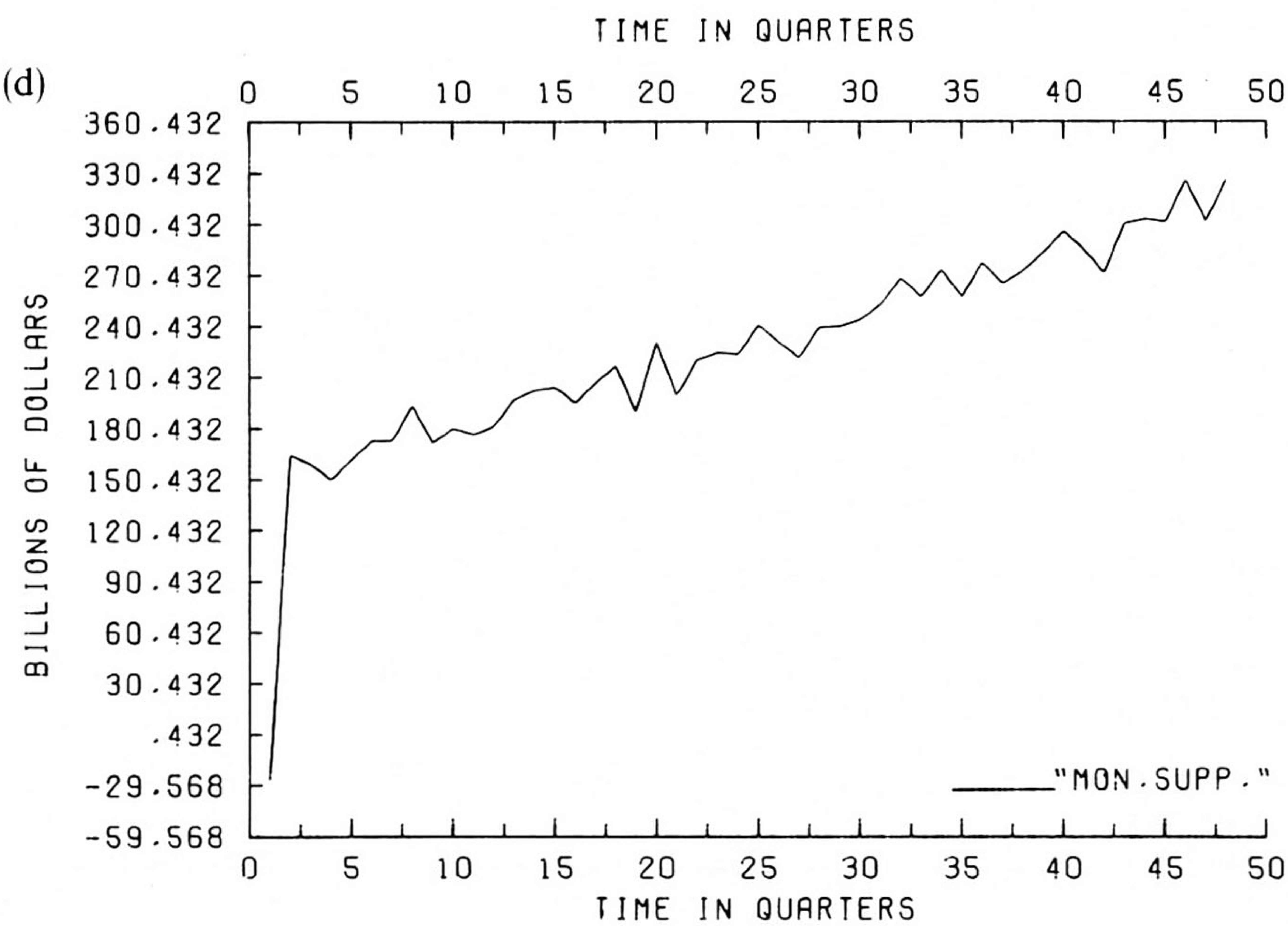


Fig. 2 (continued)

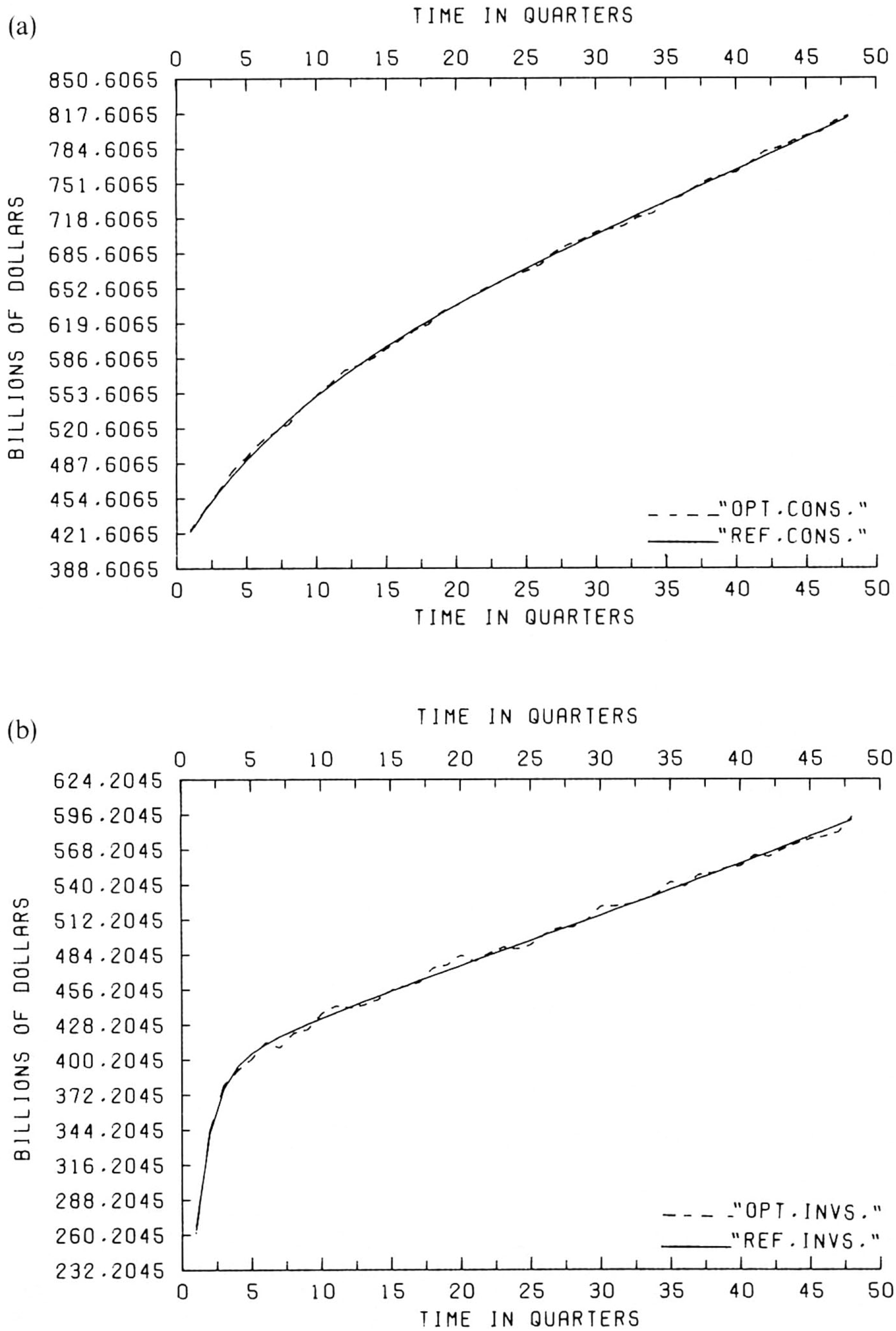
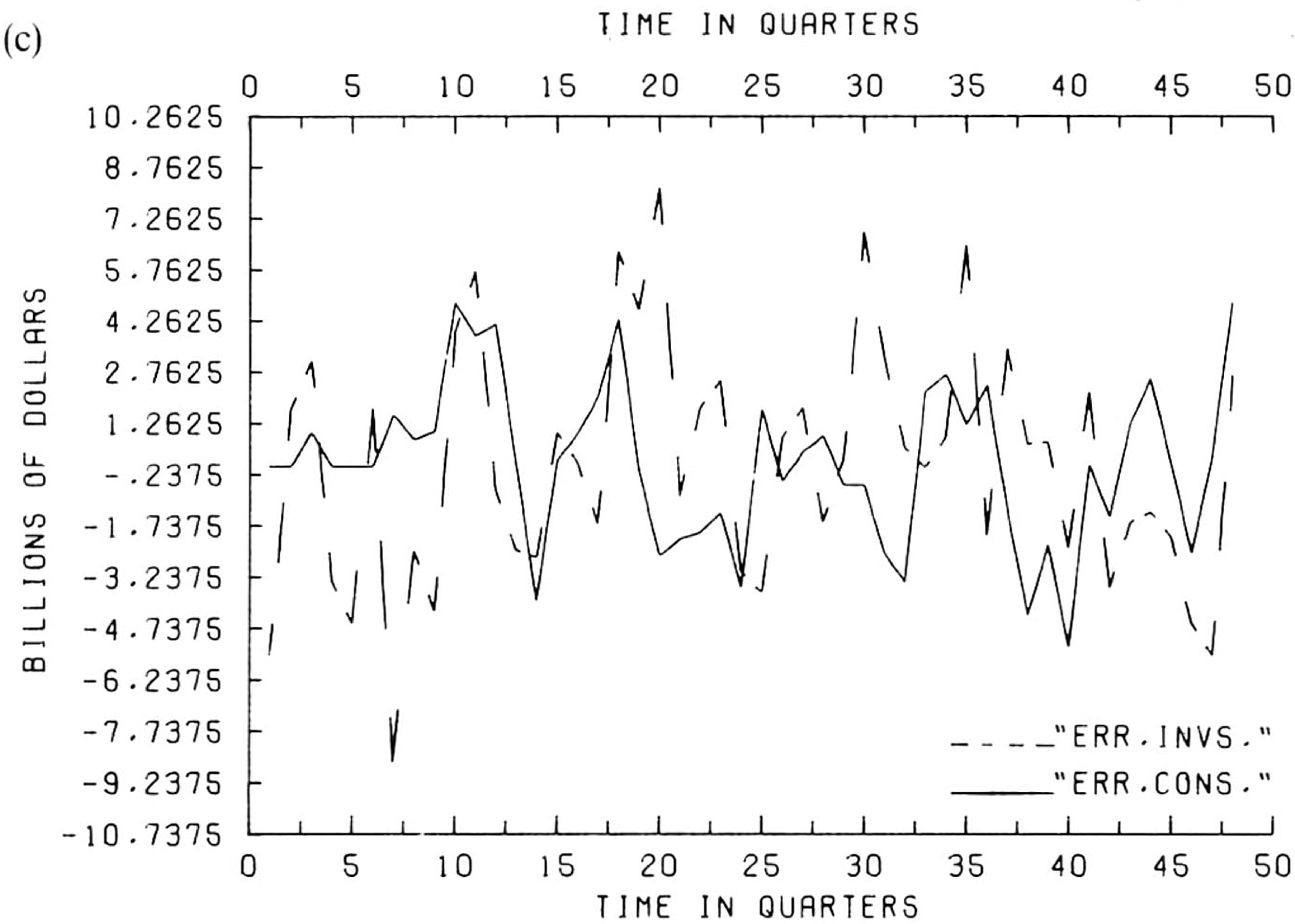


Fig. 3. Experiment 3.

(c)



(d)

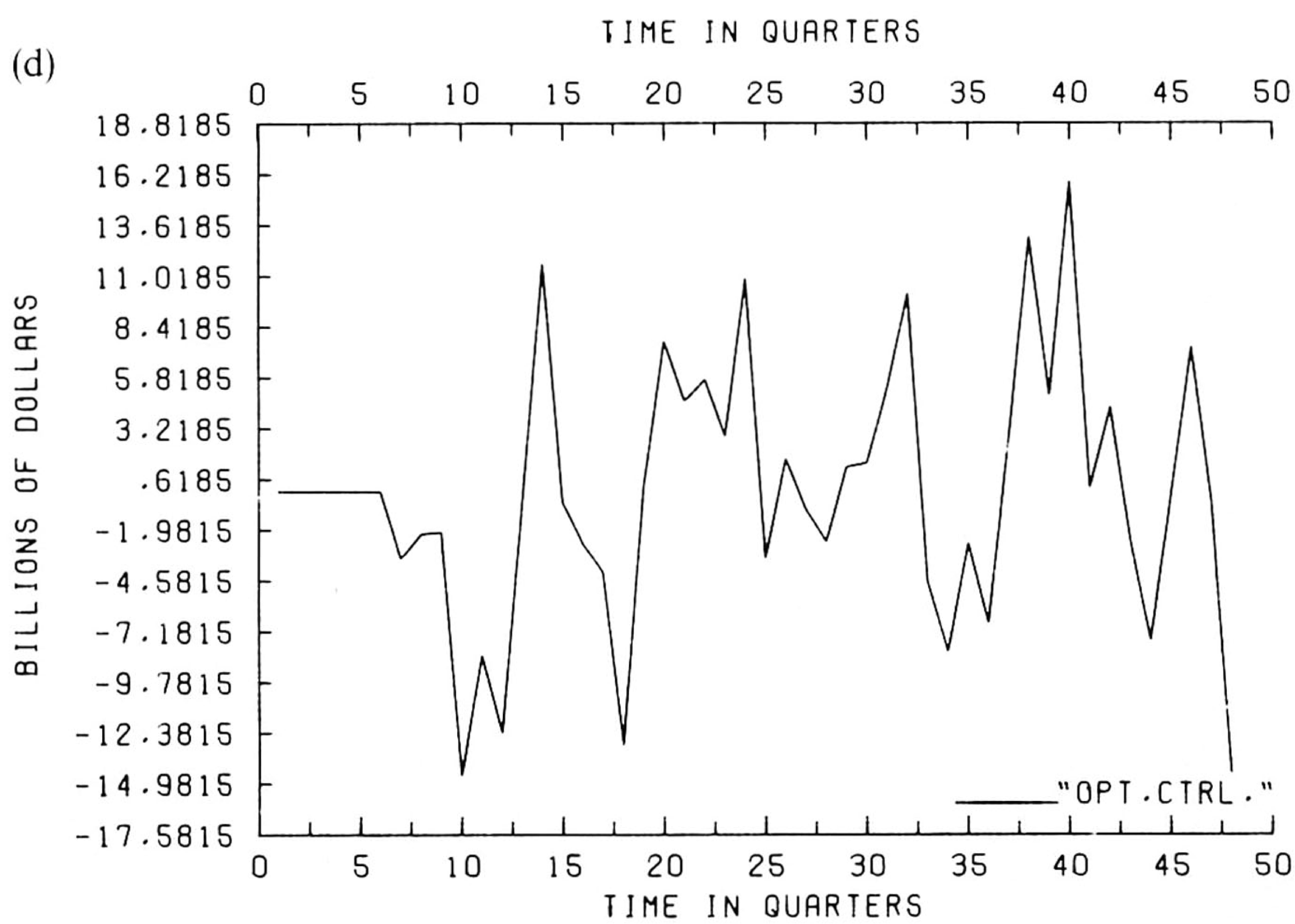
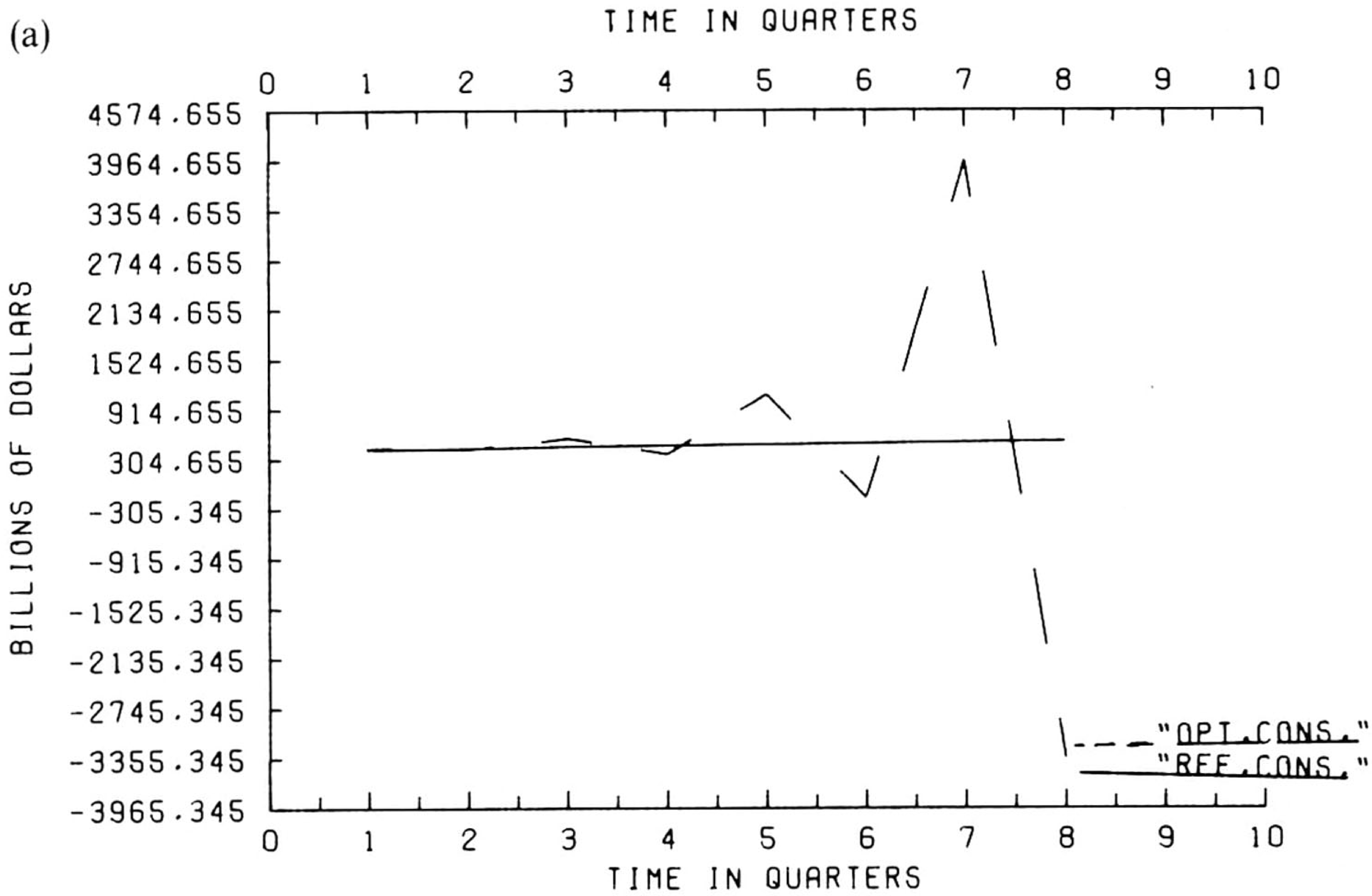


Fig. 3 (continued)

(a)



(b)

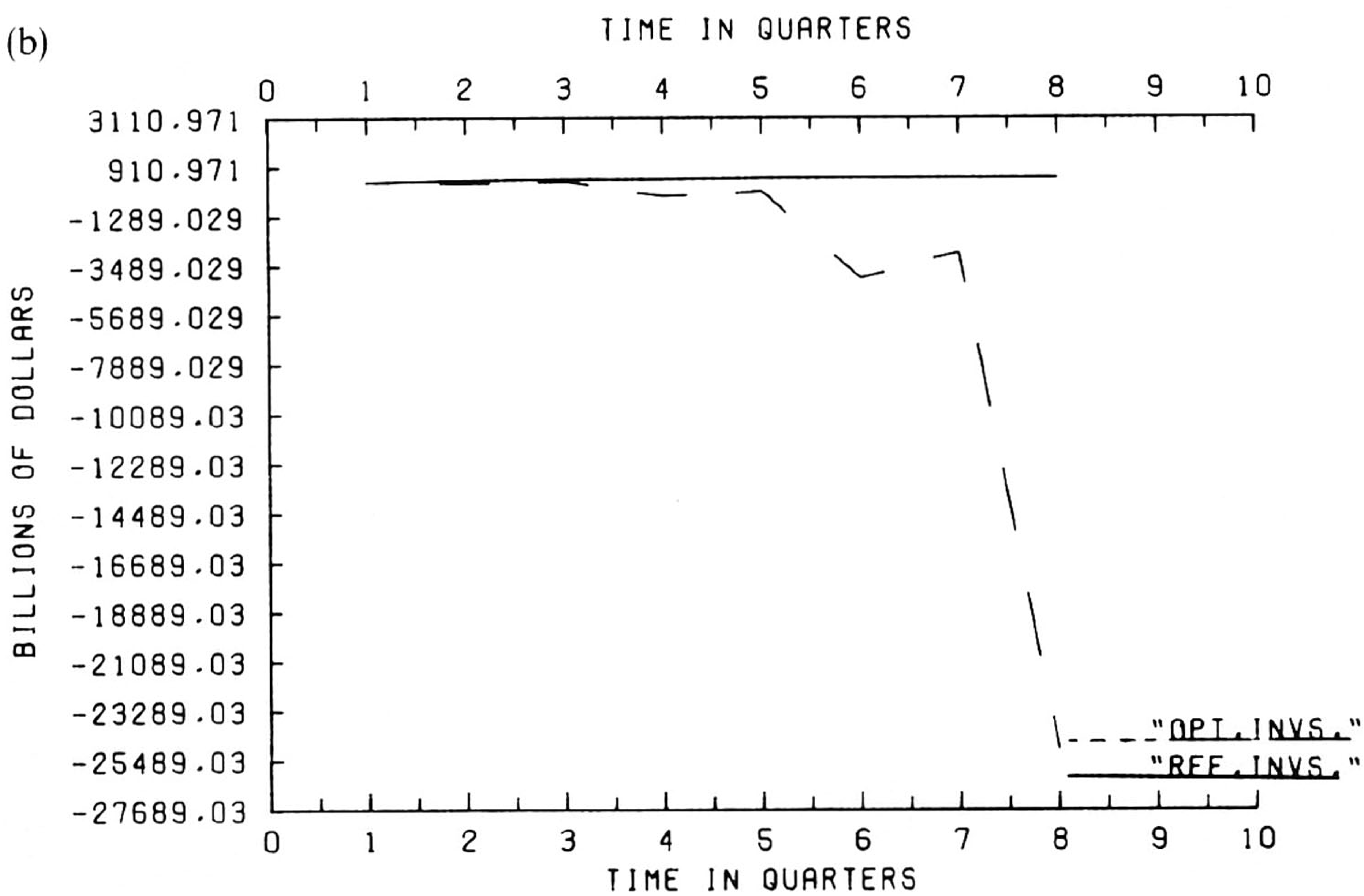
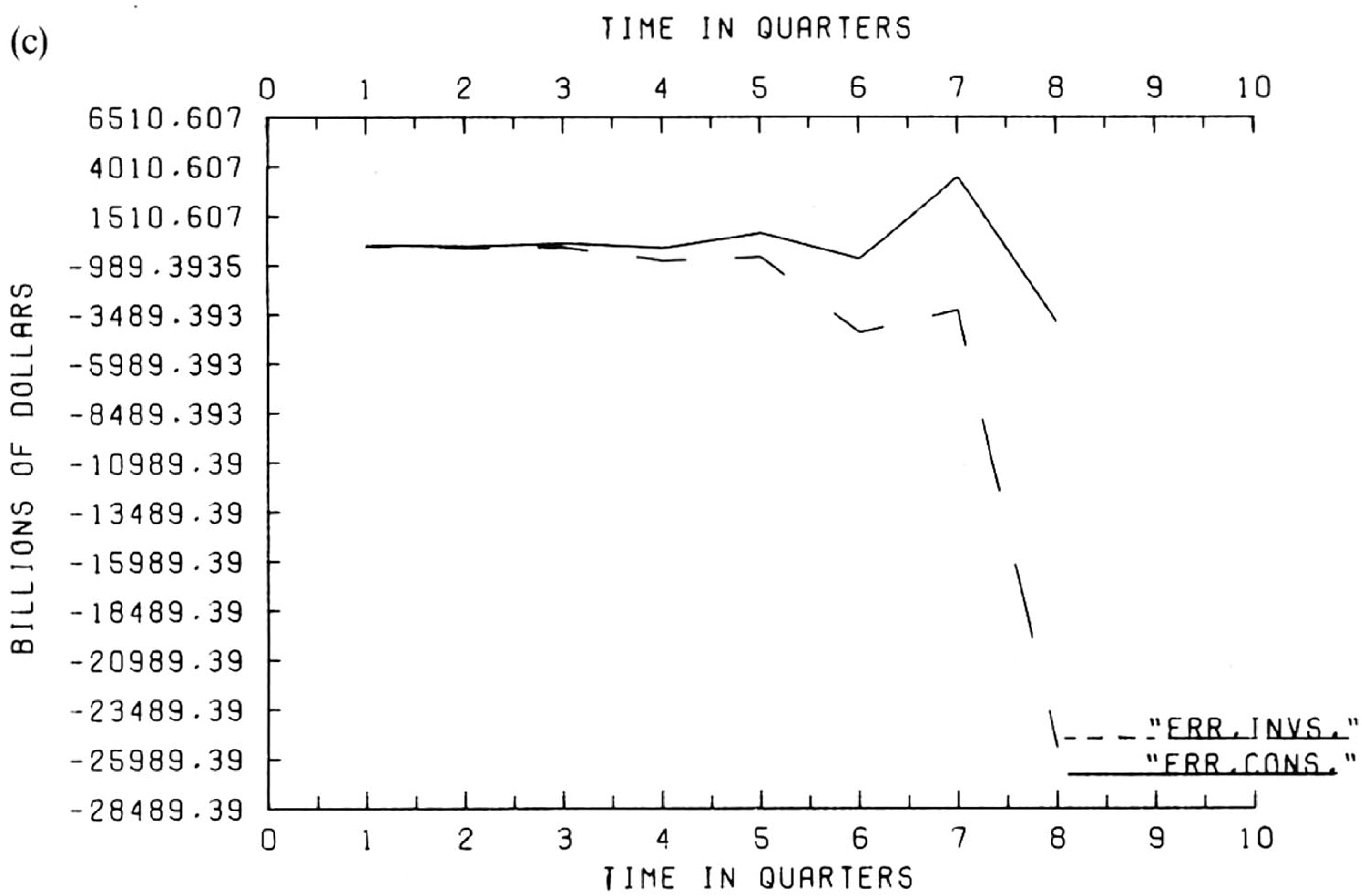


Fig. 4. Experiment 4.

(c)



(d)

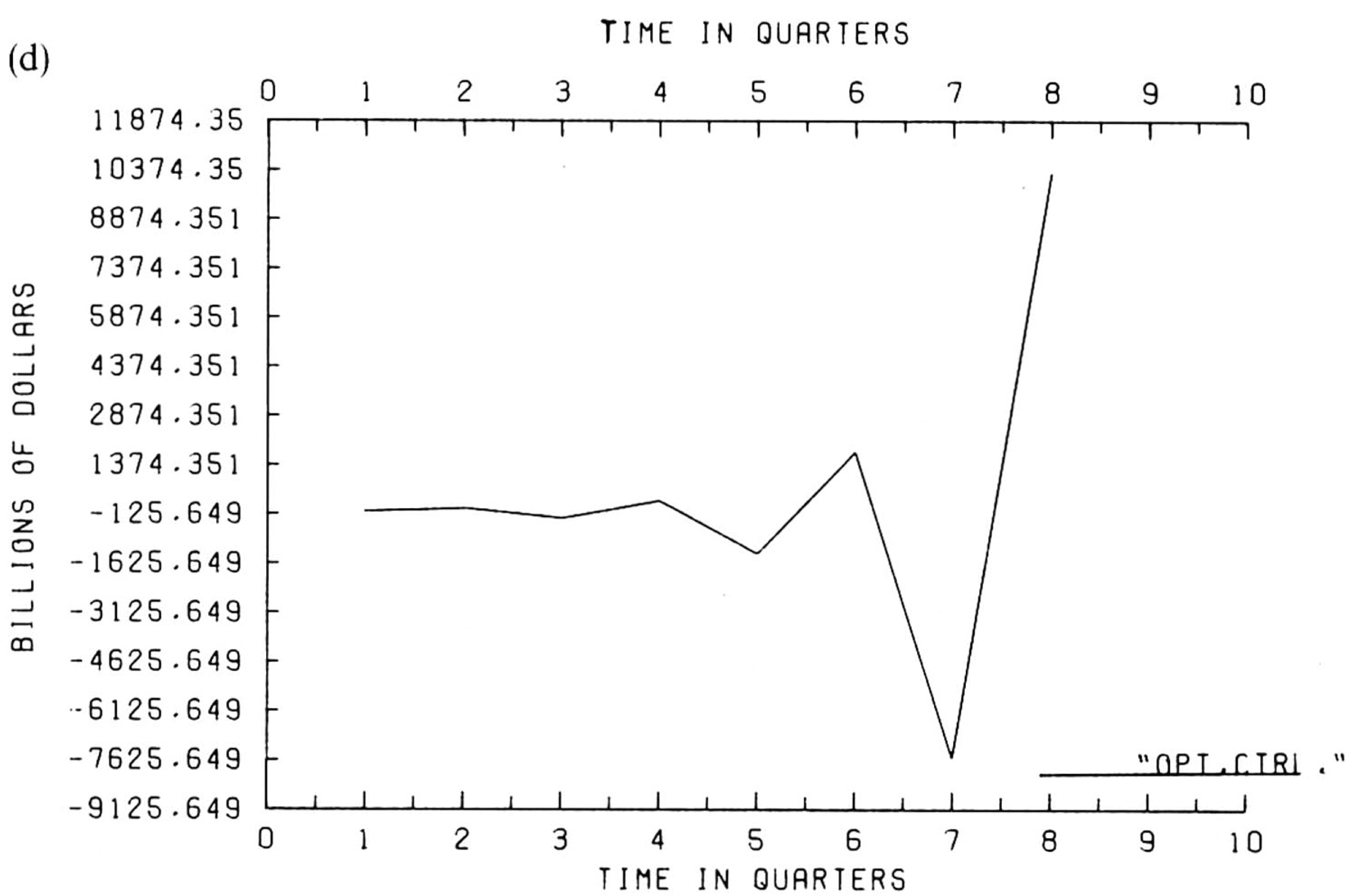


Fig. 4 (continued)

In Experiment 1 (= Construction 4) we use $Q = I$ in the control scheme, whereas in Experiment 2 we use the weighting matrix resulting from Luenberger's canonical decomposition for this system, i.e., $Q = \begin{pmatrix} 60 & -13.9 \\ -13.9 & 3.5 \end{pmatrix}$ (= Construction 1).

For both choices of the Q matrix, the closed-loop matrix PA becomes asymptotically stable. This is also clear from the figures, for in both cases the initial command error converges to zero despite white-noise influences. Figs. 1c and 2c show, moreover, that the tracking properties of the second controller are indeed better than those of the first (minimum norm) controller. Furthermore, we see in figs. 1d and 2d that this is obtained at the expense of a controller which is more sensitive to white-noise terms, due to the large components that appear in the Q matrix. So, one can say that the weighting matrices must strike a balance between tracking speed and disturbance sensitivity of the controller.

In Experiments 3 and 4, we show that for time-varying systems the policy of tracking the command error to zero as quickly as possible (= Construction 1) does not always result in an asymptotically stable closed-loop system.

To force a time-varying model, we assume in these experiments that alternatingly the government expenditures and money supply are used to regulate the system. That is, we take

$$B(2k) := \begin{pmatrix} 0.305 & 0 \\ -0.101 & 0 \end{pmatrix}, \quad B(2k+1) := \begin{pmatrix} 0 & 0.424 \\ 0 & 1.459 \end{pmatrix}, \quad k = 0, 1, \ldots .$$

Moreover, the sign of the deterministic variables in the model equation is reversed (for purely illustrative purposes), and the target paths are chosen as

$$\begin{pmatrix} C^*(k+1) \\ I^*(k+1) \end{pmatrix} = \begin{pmatrix} 0.914 & -0.016 \\ 0.097 & 0.424 \end{pmatrix} \begin{pmatrix} C^*(k) \\ I^*(k) \end{pmatrix} + \begin{pmatrix} 59.437 \\ 184.766 \end{pmatrix},$$

with $C^*(0) = 400$ and $I^*(0) = 100$.

In fig. 3 the simulation results are shown if we use the minimum norm controller (= Construction 4) for regulating this model. We see that the reference trajectories are tracked in this case.

Fig. 4 shows the simulation results for the first ten time periods if the model is controlled at each point in time by means of the corresponding Construction 1 regulator.

In contrast with Experiment 3, now a highly unstable closed-loop system is obtained. It is easily verified that this is due to the fact that the matrix $PA(0, 2)$ is not stable.

Concluding, we can say that these four experiments support the claim that, if the choice $Q = I$ in the MV regulator gives rise to a closed-loop matrix PA

that is smaller than one in norm, then this controller is a good choice within the class of successful, disturbance-rejecting controllers.

6. Conclusion

In this paper we discussed the possibilities of controlling an economy, under the assumption that the economy is described by a linear time-varying difference equation. The discussion consisted of two parts.

First of all we investigated the admissibility of prescribed target paths. A distinction was made between three types of admissibility. First, we considered the strongly-admissible target paths. Any trajectory belonging to this set can be tracked exactly during a prespecified time interval. Secondly, we distinguished the trajectories that can be tracked exactly up to an error not exceeding α over some prespecified time interval; these were called approximately-admissible target paths at level α. Finally, we discussed the trajectories that can be ultimately tracked. We called them asymptotically-admissible.

Apart from giving conditions on the dynamic evolution of an admissible target path, we also presented a method for checking the admissibility of a trajectory and constructed open-loop controllers which realize the desired behaviour of the system. Since these proposed open-loop controllers are rather unsuitable for practical use, we studied the target paths that can be tracked by a more appropriate controller. This was the subject of the second part of the paper. It turned out that the MV controller is able to track an important class of strongly-admissible target paths, as well as all asymptotically-admissible target paths. More in particular, we saw that in case the system is disturbed by noise, the weighting matrix in this controller plays a crucial role. If this weighting matrix is chosen such that the corresponding MV controller stabilizes the closed-loop system matrix, then again every asymptotically-admissible target path can be tracked. Therefore, we paid special attention to the construction of weighting matrices yielding a stable closed-loop system. We argued that in general the decision for choosing a certain matrix depends on the available amount of information concerning the future development of the system parameters.

Moreover, special attention was paid to the robustness and smoothness of the various controllers, since the choice for a particular controller depends on both the available future knowledge of the process and the desired tracking speed and smoothness of the desired controller.

We illustrated in a simulation study the pros and cons of various proposed choices. In particular, we saw that if the future of the structural parameters of an economy is unpredictable, then the best weighting matrix one can choose is the identity matrix.

An important open problem is how these tracking and smoothness results can be combined with requirements on the tracking speeds that are specific for each target variable.

Concluding, we might say that the paper treats various limitations that exist in achieving prescribed goals.

Appendix: Proof of Construction 3

Consider the system

$$\begin{aligned} &x(k+1) = A(k)x(k) + B(k)u(k)\,, \\ &x(0) = x_0\,. \end{aligned} \qquad (\Sigma)$$

From Engwerda (1988d, corol. 3) we have that this system can be rewritten as

$$\begin{aligned} &\begin{pmatrix} x'_1(k+1) \\ x'_2(k+1) \end{pmatrix} = \begin{pmatrix} A'_{11}(k) & A'_{12}(k) \\ 0 & A'_{22}(k) \end{pmatrix} \begin{pmatrix} x'_1(k) \\ x'_2(k) \end{pmatrix} + \begin{pmatrix} B'_1(k) \\ 0 \end{pmatrix} u(k)\,, \\ &x'(0) = (x'^{\mathrm{T}}_{10}\ x'^{\mathrm{T}}_{20})^{\mathrm{T}}\,, \end{aligned} \qquad (\Sigma')$$

where the subsystem $x'_1(k+1) = A'_{11}(k)x'_1(k) + B'_1(k)u(k)$ is reachable at any time k, and $x'(k) = T(k)x(k)$ with $T(k)$ a unitary matrix $\forall k$.

Definition. *$(A(.), B(.))$ is uniformly periodically smoothly exponentially stabilizable if for all k_0:*

(i) *there exist positive constants ε and k_1 such that for all $t > k_0$ there exists an integer $k_2(t)$ in the interval $[k_0 + (k-1)k_1,\ k_0 + kk_1]$ for which $S'[k_2 - 2k_1, k_2]S'^{\mathrm{T}}[k_2 - 2k_1, k_2] \geq \varepsilon I$. Here, $S'[k, k-N] := [B'_1(k) \mid A'_{11}(k+1, k)B'_1(k-1) \mid \ldots \mid A'_{11}(k+1, k-N+1)B'(k-N)$;*

(ii) *$x'_2(k+1) = A'_{22}(k)x'_2(k)$ is exponentially stable at k_0, i.e., there exist positive constants α and M such that $\|x'_2(k)\| \leq M\mathrm{e}^{-\alpha(k-k_0)}\|x'_{20}\|$ for any $k > k_0$;*

and the constants ε, k_1, M, and α satisfy the inequalities $\varepsilon(k) > \bar{\varepsilon} > 0$, $k_1(k) < \bar{k}_1 < \infty$, $M(k) < \bar{M} < \infty$, and $\alpha(k) > \bar{\alpha} > 0$, for some $\bar{\varepsilon}$, $\bar{k}_1$, $\bar{M}$, and $\bar{\alpha}$. ■

Next consider the minimization problem

$$\min_{u[k_0, k_0+N-1]} J_N(k_0, x_0) \quad \text{w.r.t.} \quad (\Sigma)\,,$$

where

$$J_N(k_0, x_0) := \sum_{k=k_0}^{k_0+N} \|x(k)\|^2 .$$

From Chow (1973) we have that the optimal control for this problem is

$$u_N(k) = -F_N(k)x(k) ,$$

where

$$F_N(k) = (B^{\mathrm{T}}(k)Q_N(k+1)B(k))^{-1}B^{\mathrm{T}}(k)Q_N(k+1)A(k) ,$$

and $Q_N(k)$ is given by the recurrence equation (5). Moreover, we have that the corresponding minimal cost equals $x_0^{\mathrm{T}}Q_N(k_0)x_0$.

Since (Σ) is uniformly periodically smoothly exponentially stabilizable, we can construct for any initial state x_0 [analogously to Engwerda (1988d, thm. 4b)] a control sequence such that independent of k_0, $\lim_{N\to\infty} J_N(k_0, x_0)$ remains bounded. So $\min_{u[k_0,.]} \lim_{N\to\infty} J_N(k_0, x_0)$ w.r.t. (Σ) exists.

Using Bellman's principle, we obtain in particular that $\lim_{N\to\infty} x_0^{\mathrm{T}}Q_N(k_0)x_0$ exists for any k_0, i.e., $\lim_{N\to\infty} Q_N(k_0) =: Q(k_0)$ exists.

On the other hand, it can now be shown using elementary analysis that $u(k) = -F(k)x(k)$ [where $F(k) := \lim_{N\to\infty} F_N(k)$] solves the above infinite-horizon optimal-control problem. So, in particular $x(k)$ must converge to zero if we use this control sequence, i.e., $\lim_{k\to\infty}(PA)(k_0, k) = 0$ for any k_0. ■

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